The Poincaré group and its conserved quantities in physics

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Abstract

In this exposition we derive from basic symmetry principles the conserved quantities that are important in physics: total energy, linear momentum, angular momentum, center of momentum, and parity. To do so we will construct the Poincaré group of spacetime symmetries step-by-step. We will use mathematical rigor to really convince the reader of the validity and fundamental nature of the results, but will choose forming good intuition over leaning into heavy group theory in order to keep the text accessible.

1 Introduction

In the entirety of physics it is undeniable that conserved quantities are one of the most interesting theoretical motivations as well as a vital tool in the solving of physics problems from classical mechanics to quantum mechanics to general relativity. While all physics students are quickly introduced to most of these quantities (total energy, linear momentum, angular momentum) in their first course on mechanics, their conservation properties are often just stated to be true, axiomatically, without any rigorous proof. This exposition works to fix that. Together we will derive the origin of these conserved quantities from fundamental symmetries of the underlying space of a given problem. At the same time we will not stray too far into the really heavy mathematical side in favor of developing intuition and covering a wider set of symmetry groups, which we will eventually combine into the final result of this exposition: the Poincaré group of spacetime symmetries.

We will work in the framework of quantum mechanics for ease of notation, but these results readily apply to most fields in physics, mainly to classical mechanics in the macroscopic limit and general relativity by comparing to the Killing vector formalism. For an introduction to the latter, refer to the book [1] by Sean Carrol, specifically chapter 3.8.

The exposition will first lay out the necessary background in group theory and operator mathematics. It will then work out in detail the derivation of the conserved quantity associated with rotation symmetry. It will then generalize the derivation to translation symmetry, after which it will generalize even further to include the various inversion symmetries and those symmetries introduced by special relativity when we add time into the mix. The exposition will end with a final conclusion and small discussion of charge, parity, and time-inversion symmetry in quantum field theory.

2 Group theoretical background

The mathematical foundation for what will be detailed in this exposition lies in the domain of group theory. While we will not use many pure group-theoretical techniques it is still in our best interest to at least get acquainted with the basic concepts, in order to respect how fundamental the final results are.

To start of, we want to define what exactly a group is.

Definition 2.1 (Group)**.** A non-empty set G is a group if for all $a, b, c \in G$: (i) G has a multiplication rule, denoted "$\cdot$", that combines a and b such that $a \cdot b \in G$; (ii) The multiplication rule is associative: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$; (iii) There exists an identity element $e \in G$: $e \cdot a = a \cdot e = a$; (iv) For each $a \in G$ there exists an inverse element $a^{-1} \in G$: $a \cdot a^{-1} = a^{-1} \cdot a = e$.

While this definition might not directly ring a bell, there are many common examples of groups. The rest of this section lists some groups we will use in the rest of the exposition.

Definition 2.2 (The cyclic group C_n)**.** The cyclic group C_n consists of elements $\{e, a, a^2, ..., a^{n-1}\}$, where e is the identity element and a is chosen such that $a^n = e$; with the canonical multiplication rule, meaning if G contains real numbers, then "$\cdot$" could just be standard multiplication, or if G contains matrices "$\cdot$" is matrix multiplication.

Proof. To prove C_n is indeed a group we see that it has an identity element and the canonical multiplication rule is associative. To see that each element has an inverse, notice how for each $a \in G$ there exist $a^{n-1} \in G$ and $a \cdot a^{n-1} = a^n = e$. Thus we may conclude C_n is indeed a group. □

The most intuitive representations of C_n can be given for C_2 and C_4. For C_2 simply say $e = 1$, $a = -1$, then naturally $a^2 = -1 \cdot -1 = 1 = e$. For C_4 say $e = 1$, $a = i$, $a^2 = -1$, $a^3 = -i$, and $a^4 = 1 = e$. The reader is advised to make sure they understand these basic examples.

Definition 2.3 (Space-inversion group I_1)**.** The space-inversion group in one dimension, I_1, constitutes a group consisting of two elements: "leaving the space as is" (e; $x = x$) or "inverting the space" (a; $x = -x$).

The group I_1 is "isomorphic" to C_2, meaning we can map each element of each group to one specific element in the other group. The reader is encouraged to perform this mapping.

Definition 2.4 (The rotation group R_1)**.** The rotation group in one parameter, R_1, constitutes all rotations in two dimensional space. There are an infinite amount of elements, one for each possible rotation angle: R_1 is a continuous group. The group elements are often denoted $R(\theta)$ or $R(\phi)$.

Proof. The multiplication rule, $R(\theta) \cdot R(\phi) = R(\theta + \phi)$, is addition of angles, which is associative. The identity element is $R(0)$ and the inverse of $R(\theta)$ is $R(-\theta)$. □

Definition 2.5 (Translation group T_1)**.** The translation group in one dimension, T_1, constitutes all translations along a line. There are an infinite amount of elements, one for each translation step: T_1 is a continuous group. The group elements are often denoted $T(x)$, $T(y)$, or $T(z)$.

Proof. The multiplication rule, $T(x_1) \cdot T(x_2) = T(x_1 + x_2)$, is addition of translation steps, which is associative. The identity element is $T(0)$ and the inverse of $T(x)$ is $T(-x)$. □

These last three groups I_1, R_1, and T_1 deserve particular attention as they will illustrate the main argument carried out in this exposition. All three characterize particular symmetries a space we wish to study may exhibit.

Space inversion, or formally "parity inversion", is a symmetry that is used to characterize whether a quantity is "odd", meaning it changes sign, or "even", meaning it does not change sign under space inversion. Examples of odd quantities are: position, velocity, acceleration, and helicity. Examples of even quantities are: mass, energy, and charge.

Rotation is a symmetry that is used to characterize whether a quantity or system is invariant under rotation of the entire space. Notable examples of such invariant systems are those that exhibit spherical symmetry, such as the ground state of the hydrogen atom or the gravitational potential around a star or planet. Such systems are often easier to describe by eliminating some of the degrees of freedom using rotation symmetry.

Translation is a symmetry that is used to characterize whether a quantity or system is invariant under linear translation of the entire space. Notable systems of such translation are crystal lattice structures, where atoms are placed in a regular pattern. Such systems are often easier to describe as a whole by applying translation symmetry to the description of just one atom.

3 Hermitian and unitary operators

Before we can move on to applications of these symmetry groups it is necessary to first lay the groundwork by discussing operators. In quantum mechanics, Hermitian and unitary operators are fundamentally interrelated with physical observables. This will become apparent by the way they interact with the Hamiltonian, which is an operator that corresponds to the total energy of a system. In this section we will develop these connections and underpin them with proofs. For the purpose of this exposition it is sufficient to think of operators as linear maps, expressed as either matrices or differentials.

3.1 Hermitian operators

The first type of operator we consider is the Hermitian operator. To define this operator we first define the Hermitian adjoint.

Definition 3.2 (Hermitian adjoint)**.** Let ψ_A and ψ_B be two wave-functions and $\hat{F}$ a linear operator. The Hermitian adjoint represented by † is defined to satisfy $\langle \hat{F}\psi_A \,|\, \psi_B\rangle = \langle \psi_A \,|\, \hat{F}^\dagger \psi_B\rangle$. If $\hat{F}$ is a matrix, $\hat{F}^\dagger$ is the complex-conjugate transpose matrix.

Definition 3.3 (Hermitian operator)**.** A linear operator $\hat{F}$ is Hermitian if $\hat{F} = \hat{F}^\dagger$.

Hermitian operators are significant since their eigenvalues represent real observables, build on the fact that all their eigenvalues are real.

Proposition 3.4. *All eigenvalues n of a Hermitian operator acting on a set of eigenfunctions ψ_n, given by the relation $\hat{F}|\psi_n\rangle = n|\psi_n\rangle$, are real.*

Proof. Notice that from the properties of the inner product $\langle \cdot \,|\, \cdot \rangle$, ψ_n being an eigenvector, and $\hat{F}$ being Hermitian we get

$$n\langle \psi_n \,|\, \psi_n\rangle = \langle n\psi_n \,|\, \psi_n\rangle = \langle \hat{F}\psi_n \,|\, \psi_n\rangle = \langle \psi_n \,|\, \hat{F}\psi_n\rangle = \langle \hat{F}\psi_n \,|\, \psi_n\rangle^* = \langle n\psi_n \,|\, \psi_n\rangle^* = n^*\langle \psi_n \,|\, \psi_n\rangle,$$

where $*$ represents complex conjugation. This implies $n = n^*$, so n must be real. □

The next important property of Hermitian operators that we will use is the following.

Proposition 3.5. *Any Hermitian operator $\hat{F}$ commutes with the Hamiltonian $\hat{H}$, so $[\hat{F}, \hat{H}] = 0$.*

We will not prove this since it is often taken as one of the postulates of quantum mechanics that is assumed to be true and supported by physical evidence. The consequence of proposition 3.5 is the following.

Proposition 3.6. *The physical quantities of an Hermitian operator $\hat{F}$ are conserved.*

Proof. To prove this we take for granted the generalized Ehrenfest theorem,

$$\frac{d\hat{F}}{dt} = -\frac{i}{\hbar}[\hat{F}, \hat{H}], \tag{3.1}$$

the derivation of which can be found in any good quantum mechanics textbook, for example in the book [2] by David J. Griffiths, chapter 3.5.3. From equation 3.1 and proposition 3.5 it naturally follows that if $\hat{F}$ is Hermitian then $\frac{d\hat{F}}{dt} = 0$, so $\hat{F}$ is conserved. □

3.7 Unitary operators

The second type of operator we consider is the unitary operator.

Definition 3.8 (Unitary operator)**.** A linear operator $\hat{U}$ is unitary if $\hat{U}\hat{U}^\dagger = \mathbb{1}$.

We naturally think of unitary operators as unitary matrices, whose determinant is one. This gives a good picture of unitary transformations of a vector-space or wave-function.

Proposition 3.9. *Unitary transformations on spaces are defined by the action of a unitary operator on the space. These transformations preserve length, area, and volume.*

Proof. Preservation of length, area, and volume corresponds (for all spaces we will consider) to the transformations having unit determinant, the defining feature of unitary transformations. □

Unitary transformations are directly tied to symmetries of the space, since they leave important defining features intact, as we shall see more of later. For now there are two more crucial properties left to show.

Proposition 3.10. *The operator $\hat{U}$ defined as $\hat{U}(\zeta) \equiv e^{-i\zeta\hat{F}}$, for some operator $\hat{F}$ and parameter ζ, is unitary.*

Proof. By Taylor series we find that for $\zeta \ll 1$ (which is the scenario of interest as we shall see later) it follows that $\hat{U}(\zeta) \equiv e^{-i\zeta\hat{F}} = \sqrt{\mathbb{1} - i\zeta\hat{F}}$, such that $\hat{U}\hat{U}^\dagger = \sqrt{(\mathbb{1} - i\zeta\hat{F})(\mathbb{1} + i\zeta\hat{F})} = \sqrt{\mathbb{1}^2 + \zeta^2\hat{F}^2} = \mathbb{1}$ for $\zeta \ll 1$. □

Proposition 3.11. *The operator $\hat{F}$ in proposition 3.10 must be Hermitian.*

Proof. This proof is not entirely rigorous but holds in all the situations we will consider. Since

$$UU^\dagger = e^{-i\zeta\hat{F}} \cdot e^{i\zeta\hat{F}^\dagger} = e^{i\zeta(\hat{F}^\dagger - \hat{F})},$$

but also

$$UU^\dagger = \mathbb{1} = U^\dagger U = e^{i\zeta\hat{F}^\dagger} \cdot e^{-i\zeta\hat{F}} = e^{i\zeta(\hat{F} - \hat{F}^\dagger)}.$$

This means $e^{i\zeta(\hat{F}^\dagger - \hat{F})} = e^{i\zeta(\hat{F} - \hat{F}^\dagger)} = \mathbb{1}$, which is only possible if $\hat{F} = \hat{F}^\dagger$, so $\hat{F}$ is Hermitian. □

4 The physical conserved quantity for R_1

We are now ready to discover our first physical conserved quantity. To represent the group R_1 we will consider the unitary transformation of rotation in two dimensions: $\hat{R}(\phi)$. It is easy to see that this transformation is unitary if we consider the familiar matrix representation

$$\hat{R}(\phi) \equiv \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}, \text{ with } \det[\hat{R}(\phi)] = \cos^2\phi + \sin^2\phi = 1. \tag{4.1}$$

Since $\hat{R}(\phi)$ is unitary we can express it as

$$\hat{R}(\phi) \equiv e^{-i\phi\hat{J}}, \tag{4.2}$$

for some Hermitian operator $\hat{J}$ called the generator of the rotation transformation. To find $\hat{J}$ we consider an infinitesimal rotation $\hat{R}(d\phi)$. If we Taylor expand $\hat{R}(\phi)$, given in equations 4.1 and 4.2, for $\phi \ll 1$ we get

$$\hat{R}(d\phi) \equiv \begin{pmatrix} 1 & -d\phi \\ d\phi & 1 \end{pmatrix} \equiv \mathbb{1} - id\phi\hat{J} \tag{4.3}$$

Solving for $\hat{J}$ gives

$$\hat{J} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{4.4}$$

Great! We are done, right? Unfortunately not. Consider the following property of $\hat{R}(\phi)$. Rotation by $\hat{R}(\phi)$ with $\phi \in [0, 2\pi)$ will rotate our entire space once, but rotation by $\hat{R}(2\phi)$ with $\phi \in [0, 2\pi)$ will rotate our entire space twice. In general $\hat{R}(m\phi)$ with $\phi \in [0, 2\pi)$ will rotate our entire space m times, but each integer value of m is an equally valid way to represent rotation of the space, since we can still orient the space in all the same way using $\hat{R}(m\phi)$ as with $\hat{R}(\phi)$. This suggests that we should refine equation 4.2 to be

$$\hat{R}^m(\phi) \equiv \hat{R}(m\phi) = e^{-im\phi} = e^{-i\phi\hat{J}} \tag{4.5}$$

which suggests $\hat{J} = m$. What is going on? As it turns out, both the 2D representation of $\hat{R}(\phi)$ given in equation 4.1 with $\hat{J}$ as in equation 4.4, as well as the 1D representation of $\hat{R}(\phi)$ given in equation 4.5 with $\hat{J} = m$ are equally valid representations. We will continue to use the 1D representation as it fits our derivations best and is "irreducible", which stems from the group-theoretical concept of representations. Different representations of the rotation group, like the 2D representation, can be reduced into this 1D representation. In a way, the 1D representation can be viewed as the most fundamental.

Using our chosen representation we are now in a place where we can examine the effect of $\hat{J}$ and $\hat{R}^m(\phi)$ on a set of eigenvectors $|m\rangle$ of the system under consideration. It is good to reflect for a moment on what these eigenvectors represent and why they are characterized by m. Our $|m\rangle$ are simultaneous eigenvectors of $\hat{J}$, $\hat{R}^m(\phi)$, and the Hamiltonian of the system. This means that any given wave-function $|\psi\rangle$ can be decomposed into these eigenvectors. This in term means that we can examine the effect of $\hat{J}$ and $\hat{R}^m(\phi)$ on ψ by examining the effect on all $|m\rangle$. This is exactly what we will do in the following section. In addition, the fact that $|m\rangle$ are eigenvectors of the Hamiltonian means that any eigenvalues associated with $|m\rangle$ are conserved physical quantities.

The action of $\hat{J}$ and $\hat{R}^m(\phi)$ on $|m\rangle$ are exactly what we would expect from the previous sections.

$$\hat{J}|m\rangle = m|m\rangle. \tag{4.6}$$

$$\hat{R}^m(\phi)|m\rangle = e^{-im\phi}|m\rangle. \tag{4.7}$$

In addition, we can get a rotated state of the system $|\phi\rangle$ from an unrotated state $|\phi_0\rangle$ by

$$|\phi\rangle = |\phi_0 + \phi\rangle = \hat{R}^m(\phi)|\phi_0\rangle. \tag{4.8}$$

We can then get a rotated wave-function $\psi(\phi)$ from and unrotated wave-function $|\psi\rangle$ by projecting the unrotated wave-function onto the rotated state

$$\psi(\phi) = \langle \phi \,|\, \psi \rangle. \tag{4.9}$$

If we expand the unrotated state into the eigenvectors $|m\rangle$, we get

$$|\phi_0\rangle = \sum_m |m\rangle. \tag{4.10}$$

We now use equation 4.8 to rotate this state and equation 4.7 to rotate each $|m\rangle$, so we get

$$|\phi\rangle = \hat{R}^m(\phi)|\phi_0\rangle = \sum_m \hat{R}^m(\phi)|m\rangle = \sum_m e^{-im\phi}|m\rangle. \tag{4.11}$$

This allows us to examine the effect of $\hat{J}$ on $|\phi\rangle$. Using equation 4.6 we find

$$\hat{J}|\phi\rangle = \sum_m \hat{J}e^{-im\phi}|m\rangle = \sum_m me^{-im\phi}|m\rangle = \sum_m i\frac{d}{d\phi}e^{-im\phi}|m\rangle = i\frac{d}{d\phi}|\phi\rangle, \tag{4.12}$$

where at the end we used $i = -1/i$ and $\frac{d}{d\phi}e^{-im\phi} = -ime^{-im\phi}$. Flipping $\hat{J}|\phi\rangle$ to $\langle\phi|\hat{J}$ by complex conjugation we can finally examine the action of $\hat{J}$ on the rotated wave-function $\psi(\phi)$. This results in

$$\langle\phi|\hat{J}|\psi\rangle = \langle \hat{J}\phi \,|\, \psi\rangle = -i\frac{d}{d\phi}\langle \phi \,|\, \psi\rangle = -i\frac{d}{d\phi}\psi(\phi). \tag{4.13}$$

This is (in units of $\hbar$) exactly the angular momentum operator we know from quantum mechanics, with eigenvalue m the projection of the angular momentum on the rotation axis! This is the profound result that underlines the purely mathematical origin of conservation of angular momentum. Notice how no physics was used in this derivation. Conservation of angular momentum is entirely a consequence of the rotational symmetry of space, and conversely, angular momentum is the generator of rotation symmetry. This profound duality is what will be explored in its fullest extend in the rest of this exposition, starting with translation symmetry.

5 The physical conserved quantity for T_1

The derivation for translation symmetry follows the same steps as rotation symmetry, so we will only show the major steps and we refer to the book [3], chapter 6, for the detailed derivation.

We represent the group T_1 as the unitary translation operator $\hat{T}(x)$. This unitary operator is generated from the Hermitian operator $\hat{P}$ as

$$\hat{T}(x) \equiv e^{-ix\hat{P}}. \tag{5.1}$$

The simultaneous eigenvectors of $\hat{T}(x)$, $\hat{P}$, and the Hamiltonian are $|p\rangle$ and the actions are given by

$$\hat{P}|p\rangle = p|p\rangle. \tag{5.2}$$

$$\hat{T}(x)|p\rangle = e^{-ixp}|p\rangle. \tag{5.3}$$

We get a translated state $|x\rangle$ from an untranslated state $|x_0\rangle$ by acting $\hat{T}(x)$ on it and we can decompose $|x_0\rangle$ into eigenvectors $|p\rangle$. We then find the action of $\hat{P}$ on $|x\rangle$ to be

$$\hat{P}|x\rangle = \hat{P}\hat{T}(x)|x_0\rangle = \sum_p \hat{P}\hat{T}(x)|p\rangle = \sum_p \hat{P}e^{-ixp}|p\rangle = \sum_p pe^{-ixp}|p\rangle = i\frac{d}{dx}|x\rangle. \tag{5.4}$$

Finally we find the action of $\hat{P}$ on a translated wave-function $\psi(x) = \langle x \,|\, \psi\rangle$ to be

$$\langle x|\hat{P}|\psi\rangle = \langle \hat{P}x \,|\, \psi\rangle = -i\frac{d}{dx}\langle x \,|\, \psi\rangle = -i\frac{d}{dx}\psi(x), \tag{5.5}$$

which we recognize as the linear momentum operator from quantum mechanics (in units of $\hbar$), with the eigenvalue p being the projection of linear momentum on the axis of movement. We have uncovered the duality that translation symmetry gives conservation of linear momentum, which in turn is the generator of translation symmetry.

6 The Euclidean group

At the current point we have found how the group R_1 of 2D rotation symmetry leads to conservation of angular momentum and how the group T_1 of 1D translation symmetry leads to conservation of linear momentum. Rotation in two dimensions is usually seen as rotation about one axis, hence the group is called R_1 and not R_2. We can naturally extend R_1 to the group R_3 of rotations in tree dimensions by including the rotations about all three spatial axes, usually by using Euler angles, see figure 1. It is instructive at this point for the reader to visualize how any 3D rotation can be decomposed into three rotations in three 2D planes.

An even easier to envision extension is from T_1 to T_3, the group of all translations in three spatial dimensions. It is natural to see how a translation in 3D space can be decomposed by subsequent translation along each axis.

As a perhaps unsurprising result we find that the combination of R_3 and T_3, denoted $R_3 \times T_3$, leads to conservation of all angular and linear momenta in 3D space. The group $R_3 \times T_3$ can further be combined with the final symmetry of space: reflection symmetry. A full derivation of the consequences of the reflection group is beyond the scope of this exposition, but the group of reflections of 3D space is denoted by I_3 and leads to parity conservation, which plays a big role in the allowed transitions between energy-levels in diatomic systems.

The combination $R_3 \times T_3 \times I_3$ is denoted E_3 and is called the Euclidean group of all symmetries of 3D space. With the construction of this group we have completed the entire non-relativistic spatial picture.

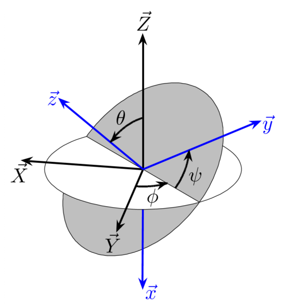

Figure 1: Rotation using Euler angles ϕ, θ, and ψ. From these three rotations we can get any rotation in 3D space. Source: [4].

7 Inclusion of time

Now that the spatial picture is finished we need figure out how to include time into the Euclidean group we have constructed. To do so we are unavoidably, but not unexpectedly, swept into the domain of special relativity. This introduces two new kinds of symmetry: time translation and Lorentz boosts. We will start with the simpler of the two.

7.1 Time translation

Translation trough time fits very naturally into the framework of spatial translation if we look at it trough the lens of special relativity. In the new theory we consider time as just another dimension and construct a 4-position as opposed to the separate 3-position and 1-time we were used to. In this way, translation through time works just like translation through space, except that instead of conserving a fourth type of momentum, we conserve total energy. This too is natural in the framework of special relativity, as the time-part of the 4-momentum is generally thought of as total energy.

7.2 Lorentz boosts

If time translation is associated with spatial translations, then it is only natural to think of Lorentz boosts as a sort of "rotation through time". This is in a way exactly the case, except that now we require that the speed of light is constant for all observers, which results in a rotation similar to the one represented by equation 4.1, but with the sine and cosine replaced by their hyperbolic counterparts. We of course immediately recognize this matrix as the one governing Lorentz transformations, indicating we have indeed made the right choices. The generator of the symmetry is a new kind of Hermitian operator, $\hat{K}$, and the conserved quantity is once more a type of momentum; this time of the center of mass. It is commonly called the "center of momentum" and its conservation means that one can always boost to an inertial frame. For a detailed derivation the reader is referred to the book [3], chapter 10. The group of all Lorentz boosts is aptly called the (improper) Lorentz group and is often denoted Λ. The proper Lorentz group $\tilde{\Lambda}$ also includes R_3.

8 The Poincaré group

We have now arrived at the penultimate section of this exposition of symmetries of 4D spacetime. Through our work we are now allowed to combine all the previous results into one group: the Poincaré group, denoted $\tilde{P}$ or $R_3 \times T_4 \times I_3 \times \Lambda$. This group contains all symmetries of 4D spacetime and from it follow all the following conserved quantities: angular momentum, linear momentum, total energy, parity, and the center of momentum. In term, these conserved quantities generate the entirety of the Poincaré group. Once more it deserves to be stressed that all these results are a purely mathematical consequence of the geometry of spacetime. Had we not known any physics, then the results would still be equally true. This gives a glimpse into just how fundamental mathematics is for describing the world around us; there truly is no other option except that these quantities are conserved.

8.1 CPT-symmetry

For completion, it is only natural to briefly mention the concept of CPT-symmetry, standing for charge-, parity-, and time-inversion symmetry. These are not commonly part of the Poincaré group; as a matter of fact, us including space-inversion is non-standard. The reason for this is that while charge-inversion symmetry and time-inversion symmetry are in a lot of theories perfectly valid, they turn out not to be in many quantum field theories. This seems highly troubling for our view of reality, but as it turns out, the situation is more complex. They are indeed conserved, but only the three of them together. What this means in terms of the mathematics is that the three of them together have one quantum-number/set of eigenvalues, so one has to include all of them or none of them, hence they are often left out of the Poincaré group.

9 Further reading

For further reading the reader is directed to the fantastic book "Group theory in physics", by Wu-Ki Tung [3], on which much of the content of this exposition is based, albeit in a very simplified form. The book introduces the reader to the entire group-theoretical background and builds the concepts in this exposition up from really the most basic principles. It also contains the detailed and exact versions of all derivations presented or referenced in this exposition.

References

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