The Standard Model as a Geometric Theory
(IN PROGRESS)

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Abstract

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0.1 Introduction

Quantum Field Theory (QFT) is one of the two most successful theories describing our reality, the other being *General Relativity (GR).* While GR is treated as a geometrical theory first and foremost, this is not the case for most introductions to QFT. This choice is easily understood, as one most naturally arrives at QFT from *single-particle Quantum Mechanics (QM)*, so it is only natural to primarily treat the "quantum" and "field" aspect of QFT. As arguably the most important product of QFT, the *Standard Model of Particle Physics (SM)* is treated much the same way. Dirac fermions are introduced as matter particles and *Yang-Mills (YM)* gauge fields host gauge bosons as force carriers. In doing so, the SM is built primarily as a theory arising from local unitary gauge symmetries. The first peek behind the curtain comes in the form of the covariant derivative. This object is in most cases introduced as a replacement to the normal derivative, with the purpose of magically fixing the Lagrangian. The Lagrangian had lost its invariance under gauge transformations now that these transformations have been made local, and the covariant derivative neatly fixes this. However, this poses the question: What is the covariant derivative really doing to undo the mess that the local gauge transformation caused? To answer this question neatly, one must zoom out and treat the SM as a geometric theory, just as GR. Doing so exposes the beautiful mathematical structure underlying the fundamental forces, as well as exposing some of the many difficulties of treating gravity as a theory of quantum fields.

A note on conventions: *Throughout the text we will use natural units. We also use Einstein sum convention, including that lower indices contract with upper indices. In this regard we are more precise when it comes to index placements, which will be favourable in the index-rich environment in which we will be working. In section 1 we will use Roman indices for group indices and Greek indices for spatial indices. Once we start on the physics, these index sets will be refined, which will be indicated at that point.*

Chapter 1

The Geometry of Reality

We describe quantum fields to be living on a *base space*, which is a smooth, oriented, Lorentzian *manifold* $\mathcal{M}$ with an associated Lorentzian metric g. The standard in QFT and the SM is that $\mathcal{M}$ encodes the geometry of flat four-dimensional *Minkowski spacetime.* Unless otherwise specified, this will be assumed throughout this text.

base space
manifold
Minkowski spacetime

1.1 Fibre bundles and Lie theory

Now consider a *topological space* F called the *fibre.* We can form a new topological space E, called the *fibre bundle*, by placing at each point $x \in \mathcal{M}$ a copy of F. We can introduce a map $\pi : E \mapsto \mathcal{M}$ such that $E_x := \pi^{-1}(x) = F$ for all $x \in \mathcal{M}$. All maps we will consider in this text will be smooth unless specified otherwise.

topological space
fibre
fibre bundle

If the fibre F is a (matrix) *Lie group* G, then we can turn the fibre bundle E into a *principal fibre bundle* P. A principal fibre bundle has the advantage of *locally* functioning as a Cartesian product space $P \cong \mathcal{M} \times G$. This requires that there is a *right action* $R : P \times G \mapsto P$ of G on P, namely $(x, g)h = (x, gh)$ for all $(x, g) \in P$ and $h \in G$. The right action for a fixed element $g \in G$ is denoted $R_g : P \mapsto P$. The map $\pi : P \mapsto \mathcal{M}$ acts as projection on the first factor $(x, g) \mapsto x$. Note that this is an entirely local description, and in fact *globally* $P \ncong \mathcal{M} \times G$. In particular, for every $x \in \mathcal{M}$ there exists a neighbourhood $\mathcal{U}$ of x such that $\pi^{-1}(\mathcal{U}) \cong \mathcal{U} \times G$.

Lie group
principal fibre bundle
right action

At in a neighbourhood $\mathcal{U}$ of each point $x \in \mathcal{M}$ we can choose an element g of the fibre G at that point, equivalent to selecting an element $(x, g) \in P$. If we do so in a smooth manner, such that the element g chosen at each x varies smoothly in G from point to point, then we specify a local *section* $\phi : \mathcal{U} \subset \mathcal{M} \mapsto P, x \mapsto (x, g)$. We require that $\pi \circ \phi = \mathrm{id}_{\mathcal{M}}$, the identity map on $\mathcal{M}$, which enforces that the section can only assign elements from the fibre $P_x = \pi^{-1}(x)$, above x, to x. If G is a gauge group, we can think of assigning an element of the gauge group to each point in our spacetime. This corresponds to the action of "making a gauge choice".

section

Associated with the Lie group G is its *Lie algebra* $\mathfrak{g}$, which is defined as the tangent space to G the identity element $e \in G$, $\mathfrak{g} := T_e G$. Since a Lie group is a manifold there exists

Lie algebra

a chart $g : \mathbb{R}^n \mapsto G, \alpha_a = (\alpha_1, \ldots, a_n) \mapsto g(\alpha) = g$, where we identify $g(\alpha)$ with the element $g \in G$ that it maps to and $\dim(G) = n$. The Lie group has a set of generators $\{\xi^a\}$ that are span the Lie algebra and are defined as $\xi^a := \frac{\partial}{\partial \alpha_a} g(\alpha)|_{\alpha=0}$. All Lie groups we consider will be *compact* and *connected*, which means that any group element is recovered through the exponential map $g(\alpha) = \exp(\alpha_a \xi^a)$, which is defined as the infinite powers series for the exponential. The internal structure of the Lie algebra is defined through the *Lie bracket* $[\cdot, \cdot]$ by relating generators according to structure constants ${f^{ab}}_c$ as: $[\xi^a, \xi^b] = {f^{ab}}_c \xi^c$.

compact
connected
Lie bracket

Since the Lie algebra is the tangent space of the fibre, an element $\mathfrak{X} = \alpha_a \xi^a \in \mathfrak{g}$ is a vector in this tangent space. However, this does not immediately make $\mathfrak{X}$ an element of the tangent space T_pP at a point $p \in P$. In order to make this happen we build a vector field on P using the exponential map. The element $\mathfrak{X}$ acts on P at a point p with the exponential map as $p \mapsto p\exp(t\mathfrak{X})$. This defines an *integral curve* $\gamma : t \mapsto p\exp(t\mathfrak{X})$ in P starting at p ($\gamma(0) = p$) along the vector field generated by $\mathfrak{X}$. To find the vector field we just need to find the tangent vector to this integral curve at each point p. This produces a vector field $\mathfrak{X}^{\#}(p) := \frac{d}{dt}|_{t=0} p\exp(t\mathfrak{X})$ on P such that indeed $\mathfrak{X}^{\#}(p) \in T_pP$. In terms of the right action the integral curve is the mapping $t \mapsto R_{\exp(t\mathfrak{X})}(p)$ such that $\mathfrak{X}^{\#}(p) = \frac{d}{dt}|_{t=0} R_{\exp(t\mathfrak{X})}(p)$.

integral curve

There are two important mappings that will come up multiple times in what follows. Consider a map $f : \mathcal{M} \mapsto \mathcal{N}$ between manifolds. At a point $x \in \mathcal{M}$ the *pushforward* $(df)_x : T_x\mathcal{M} \mapsto T_{f(x)}\mathcal{N}$ acts on a tangent vector $X \in T_x\mathcal{M}$ and a test function $\varphi \in C^\infty(\mathcal{N})$ as $(df)_x(X)[\varphi] = X[\varphi \circ f]$.

pushforward

As an example, consider the map $\pi : P \mapsto \mathcal{M}$ such that $\pi^{-1}(x) = P_x \cong G$. In the same way we can define a map $d\pi_p : T_pP \mapsto T_x\mathcal{M}$ as a map between tangent spaces. This is exactly the pushforward of π at p. As the notation suggest, the pushforward generalizes the notion of total differentiation from ordinary calculus.

As another example we consider the pushforward of the right action $R : P \times G \mapsto P$. Consider the right action $R_g : P \mapsto P$ for a fixed group element $g \in G$. The pushforward is $(dR_g)_p : T_pP \mapsto T_{pg}P$, which maps tangent vectors at one point along the fibre to a tangent vector at another point along the fibre.

The second mapping is the *pullback* by f, denoted f^*. It is used to pull back a tensor field from $\mathcal{N}$ to $\mathcal{M}$. Important to our application, the pullback of a k-form ω at $f(x) \in \mathcal{N}$ to $x \in \mathcal{M}$ is defined as $(f^*\omega)_x(\mathcal{X}_1, \ldots, \mathcal{X}_k) = \omega_{f(x)}(df_x(\mathcal{X}_1), \ldots, df_x(\mathcal{X}_k))$, for $\mathcal{X}_j \in T_x\mathcal{M}$. The pullback generalizes the notion of function precomposition. Important properties of the pullback are compatibility with the wedge product, $f^*(\omega \wedge \omega') = f^*\omega \wedge f^*\omega'$, and compatibility with the exterior derivative, $f^*(d\omega) = d(f^*\omega)$.

pullback

1.2 The Connection

We may now consider what the tangent space to the principal bundle looks like. Since for each $x \in \mathcal{M}$ there exists a neighbourhood $\mathcal{U}$ of x such that for a *local trivialization* $P|_{\mathcal{U}} = \pi^{-1}(\mathcal{U}) \cong \mathcal{U} \times G$, we see that $T_pP|_{\mathcal{U}} \cong T_x\mathcal{M}|_{\mathcal{U}} \oplus T_gG$ for every $p = (x, g) \in P$. The term “trivialization” refers to the fact that locally there is the trivial structure of a product manifold. Conventionally we call the part associated with the tangent space to

local trivialization

the manifold at p the *horizontal subspace* $T_x\mathcal{M} \sim H_pP \subset T_pP$ and the part associated with the tangent space to the fibre at p the *vertical subspace* $T_gG \sim V_pP \subset T_pP$, such that $T_pP \cong H_pP \oplus V_pP$. It is important to stress again that this association is entirely local. There is no unique way of dividing tangent space of all points in the principal bundle into an horizontal and vertical subspace. Exactly this non-uniqueness allows for the freedom in which physics can be encoded. The principal bundle is a non-trivial product of the manifold and the fibre, such that locally it looks like a proper product space, but globally there is non-trivial geometry. In the mathematical literature this global behaviour is often described as a *twisted product* of the manifold and the fibres, $P = \bigcup_{x\in\mathcal{M}} P_x$. Viewing the principal bundle as a manifold, this twisting is encoded in the transition function $g_{\alpha\beta}(x) : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \mapsto G$ of two neighbourhoods $\mathcal{U}_\alpha$ and $\mathcal{U}_\beta$ of x. If the transition functions are non-trivial, the bundle is twisted.

horizontal subspace

vertical subspace

twisted product

Let us make this more precise. We can then define the vertical subspace directly as $V_pP := \ker d\pi_p$ and $V_pP \cong \mathfrak{g}$. There is no freedom in this definition, the vertical subspace is intrinsic to the bundle. By the *Linear First Isomorphism Theorem* we find that since $d\pi_p$ is a linear map between vector spaces $T_pP/\ker d\pi_p \cong \mathrm{im} d\pi_p$, which gives $T_pP/V_pP \cong T_x\mathcal{M}$. The horizontal subspace H_pP is a complementary subspace to V_pP. This means that $H_pP \cap V_pP = \{0\}$ and in particular that $d\pi_p|_{H_pP} : H_pP \mapsto T_x\mathcal{M}$ is an isomorphism, so $T_pP/V_pP \cong H_pP$. However, there are infinitely many such complements H_pP. As an example, consider $\mathbb{R}^2 \cong \mathbb{R} \oplus \mathbb{R} \cong H \oplus V$. Let $V = \mathrm{span}\{(0,1)\}$, then $\mathbb{R}^2/V \cong \mathbb{R} \cong H$. We find $H = \mathrm{span}\{(1,a)\}$ for any $a \in \mathbb{R}$ is a valid choice for H_pP, since then $H \cap V = \{0\}$ and $\mathbb{R}^2 = \mathrm{span}\{(1,a),(0,1)\}$. Thus, in the end, there is freedom in choosing a horizontal subspace. Let us assume this choice is made. We will later see that this is done by the principle of least action. What do the two complementary subspaces allow us to do?

Linear First Isomorphism Theorem

A vector $X(p) \in T_pP$ specifies a direction in the principal bundle. A natural question to ask is which part of $X(p)$ points along the “gauge direction” and which part points along the “spacetime direction”? We now have the means to answer this question. We can decompose $X(p)$ into a horizontal “spacetime” part $\tilde{\mathscr{X}}(p)$ and a vertical “gauge” part $\mathfrak{X}^{\#}(p)$ as

$$X(p) = \tilde{\mathscr{X}}(p) + \mathfrak{X}^{\#}(p),$$

where $\mathfrak{X}^{\#}(p) := \mathrm{proj}_{V_pP}X(p)$ and $\tilde{\mathscr{X}}(p) := \mathrm{proj}_{H_pP}X(p)$ are defined as $X(p)$ projected on each of the complementary subspaces. Since $V_pP \cong \mathfrak{g}$ we know that $\mathfrak{X}(p)$ is a Lie algebra element, from which we generate the vector $\mathfrak{X}^{\#}(p) \in V_pP$. We call the projection map onto the Lie algebra at $p \in P$ the *connection 1-form* at p and denote it as $\omega_p : T_pP \mapsto \mathfrak{g}$. It has two defining characteristics that can be formulated when turning $\mathfrak{X}^{\#}(p)$ into a vector field and ω_p into a 1-form ω on the whole principle bundle. The first is that ω extracts the Lie algebra component of the vector field it acts on, $\omega(\mathfrak{X}^{\#}) = \mathfrak{X}$. The second is that we can now define the horizontal subspace as $H_pP := \ker(\omega_p)$. This makes exact the idea that the horizontal subspace contains the vectors that point entirely into the spacetime direction, since ω filtered out the gauge direction. The connection 1-form projects out of any vector field $X(p)$ on P the change along the gauge direction.

connection 1-form

An important property of the horizontal subspace is that it is invariant under the (pushforward of) right group action R_g. This should come as no surprise, since all the information about G lives in the vertical subspace. In a sense, invariance under dR_g is the very way we defined H_pP. As an equation this means that

$$dR_g H_p P = H_{gp} P.$$

We use this to prove a fundamental property of the connection, namely its transformation law under right group action, that is, under local gauge transformations. That is, we want to show that for some $X(p) \in T_pP$ we have

$$\omega_p(X(p)) \mapsto \omega_{pg}((dR_g)X(p)) \overset{!}{=} f\big(\omega_p(X(p))\big),$$

where f gives the correct transformation law for ω. We start by deriving a general property of the vector field $\mathfrak{X}^{\#}$ generated by some Lie algebra element $\mathfrak{X} \in \mathfrak{g}$. Let $g \in G$ be an element of the Lie group and $p \in P$ a point in the principal bundle, then

$$\mathfrak{X}^{\#}(pg) = \frac{d}{dt}\Big|_{t=0} pg\exp(t\mathfrak{X}) = \frac{d}{dt}\Big|_{t=0} p\exp(tg\mathfrak{X}g^{-1})g = (dR_g)(g\mathfrak{X}g^{-1})^{\#}(p),$$

where we used the identity $g\exp(t\mathfrak{X})g^{-1} = \exp(tg\mathfrak{X}g^{-1})$. This in turn follows from $g\mathfrak{X}^n g^{-1} = (g\mathfrak{X}g^{-1})^n$, which is easy to prove by induction. By applying this result we then get

$$\omega_{pg}(\mathfrak{X}^{\#}(pg)) = \omega_{pg}((dR_g)(g^{-1}\mathfrak{X}g)^{\#}(p)) = f\big(\omega_p((g^{-1}\mathfrak{X}g)^{\#}(p))\big) = f\big((g^{-1}\mathfrak{X}g)^{\#}(p)\big) \overset{!}{=} \mathfrak{X},$$

Since $\omega(\mathfrak{X}^{\#}) = \mathfrak{X}$. Clearly, there is only one option for f, it is the inverse of the adjoint operation. We then find that

$$\omega_p \mapsto \omega_{pg} \circ dR_g = g^{-1}\omega_p g.$$

This equation shows under the right group action of G the connection 1-form ω carries the adjoint action. In particular, since $\mathfrak{X} \in \mathfrak{g}$ is a Lie algebra element, the right action of G on ω forms an adjoint representation of the Lie algebra $\mathfrak{g}$, as we shall make repeated use of later. This property is called *adjoint covariance*.

adjoint covariance

1.3 Curvature

With the connection in hand we are finally in a place where we can discuss the curvature in the gauge field. We define curvature as the distance moved in the gauge direction when moving along a closed spacetime curve. Let us make this precise. Infinitesimally, a tangent vector records distance moved in a certain direction. We wish to combine tangent vectors such that they form an infinitesimal loop in the spacetime manifold $\mathcal{M}$ and then measure the difference vector in the fibre direction (gauge direction) after travelling around the loop once.

We consider first this picture in the principal bundle P. Let X, Y be vector fields on

P and let $Z \in T_pP$ be a vector at $p \in P$. We can get the *horizontal projection* of X as $\tilde{\mathcal{X}} := X - \omega(X)^{\#}$ and in the same way we get $\tilde{\mathcal{Y}}$. The horizontal vectors point purely along the spacetime direction in P. Consider the action of transporting Z around the loop $\tilde{\mathcal{X}} \to \tilde{\mathcal{Y}} \to -\tilde{\mathcal{X}} \to -\tilde{\mathcal{Y}}$. The resulting vector is $Z' = (-\tilde{\mathcal{Y}})(-\tilde{\mathcal{X}})\tilde{\mathcal{Y}}\tilde{\mathcal{X}}Z$. The change of the Z vector around the curve should be given by the difference $Z' - Z$, but equivalently we may ask for the difference $\tilde{\mathcal{X}}\tilde{\mathcal{Y}}(Z' - Z)$, since the difference is invariant under change of the loop's starting point. We then get

horizontal projection

$$\tilde{\mathcal{X}}\tilde{\mathcal{Y}}(Z' - Z) = \tilde{\mathcal{X}}\tilde{\mathcal{Y}}Z' - \tilde{\mathcal{X}}\tilde{\mathcal{Y}}Z = \tilde{\mathcal{Y}}\tilde{\mathcal{X}}Z - \tilde{\mathcal{X}}\tilde{\mathcal{Y}}Z = -[\tilde{\mathcal{X}}, \tilde{\mathcal{Y}}]Z.$$

This tells us that we extract the difference around the loop by acting with $-[\tilde{\mathcal{X}}, \tilde{\mathcal{Y}}]$. We define the curvature as the gauge component of the difference, that is, the vertical component. This gives the definition of the *principal curvature* as $\Omega(X, Y) := -\omega([\tilde{\mathcal{X}}, \tilde{\mathcal{Y}}])$. In Appendix A it is shown that this can be written independent of the vectors X and Y as

principal curvature

$$\Omega = d\omega + \frac{1}{2}[\omega, \omega] = d\omega + \omega \wedge \omega.$$

This equation is called the *second Cartan structure equation.* It gives the curvature of the gauge field as measured on the principal bundle P. The principal curvature transforms as follows

second Cartan structure equation

$$\begin{aligned}
\Omega_{pg}(X, Y) &= d(\omega_{pg})(X, Y) + \frac{1}{2}[\omega_{pg}(X), \omega_{pg}(Y)] \\
&= g^{-1}d(\omega_p)(X, Y)g + \frac{1}{2}(g^{-1}\omega_p(X)gg^{-1}\omega_p(Y)g - g^{-1}\omega_p(Y)gg^{-1}\omega_p(X)g) \\
&= g^{-1}\Big(d(\omega_p)(X, Y) + \frac{1}{2}[\omega_{pg}(X), \omega_{pg}(Y)]\Big)g = g^{-1}\Omega_p(X, Y)g.
\end{aligned}$$

This of course means that Ω also carries the adjoint action and is an adjoint covariant object, just like the connection.

1.4 Objects on spacetime

Everything we have considered up until has been done in the principal bundle P. We now wish to represent these concepts on the base manifold $\mathcal{M}$, as this is where the physics lives. Let us pick a local section $\phi : \mathcal{U} \subset \mathcal{M} \mapsto P$, for $\mathcal{U}$ a neighbourhood of some $x \in \mathcal{M}$. We can use the pullback ϕ^* of the local section to pull objects on P onto $\mathcal{U} \subset \mathcal{M}$. In doing so we can start connecting objects on P to the objects that show up in physics.

1.4.1 The Gauge Potential

We start with the Lie algebra valued connection 1-form ω. On the base manifold, that is, on our spacetime manifold, this gives the *gauge potential* $\mathcal{A} := \phi^*\omega$, which is a Lie algebra valued 1-form on spacetime. At each spacetime point we can choose a Lie group element to create a map $g(x) : \mathcal{U} \mapsto G$. The section $\phi(x)$ selected a point of the fibre P_x.

gauge potential

We can now move along P_x by acting with $g(x)$ such that we transform to a new section $\phi'(x) := \phi(x)g(x)$. This procedure is known as a *local gauge transformation*. The local gauge transformation gives a right group action at each point $p \in P$, which we represent with a map $R : P \times G \mapsto P, R(p,g) = pg$, where we remember that $(x,h)g = (x,hg)$ for $x \in \mathcal{M}$ and $h \in G$. We investigate how $\mathcal{A}$ transforms under a local gauge transformation $\phi(x) \mapsto \phi'(x) = R(\phi(x), g(x))$. Let $\mathcal{X} \in T_x\mathcal{M}$ such that we find by the definition of the pullback that

local gauge transformation

$$\mathcal{A}_x(\mathcal{X}) \mapsto \mathcal{A}'_x(\mathcal{X}) = \omega_{\phi'(x)}(d\phi'(\mathcal{X})).$$

By using the chain rule and using that the differential splits linearly we find

$$d\phi'_x(\mathcal{X}) = dR_{\phi g}(d\phi_x(\mathcal{X}), dg_x(\mathcal{X})) = dR_{\phi g}(d\phi_x(\mathcal{X}), 0) + dR_{\phi g}(0, dg_x(\mathcal{X})).$$

The argument $d\phi_x(\mathcal{X})$ in the first term is an entirely horizontal vector, while $dg_x(\mathcal{X})$ in the second term is an entirely vertical vector. Applying the covariance of ω to the first term gives

$$\omega_{\phi'(x)}\big(dR_{\phi g}(d\phi_x(\mathcal{X}), 0)\big) = g^{-1}(x)\omega_{\phi(x)}\big(d\phi_x(\mathcal{X})\big)g(x) = g^{-1}(x)\mathcal{A}_x(\mathcal{X})g(x).$$

In the second term $dg_x(\mathcal{X})$ lives in T_gG. Since $\mathcal{A}'_x(\mathcal{X})$ should be Lie algebra valued, $\omega_{\phi'(x)}$ should return an element of the Lie algebra. To do so $dR_{\phi g}(0, dg_x(\mathcal{X}))$ should be some vector field $\mathfrak{X}^{\#}$ generated by a Lie algebra element $\mathfrak{X}$, such that $\omega_{\phi'(x)}(\mathfrak{X}^{\#}) = \mathfrak{X} \in \mathfrak{g}$. This means that we should extract from $dg_x(X)$ the element $\mathfrak{X}$ that generated the curve along $\mathfrak{X}^{\#}$ that eventually has tangent vector $dg_x(\mathcal{X})$. This is easily done by acting with the inverse group element to give $\mathfrak{X} = g^{-1}(x)(dg_x(\mathcal{X})) \in \mathfrak{g}$. This object is the *Maurer-Cartan form* for matrix groups. Combining the first and second term, and dropping the dependence on $\mathcal{X}$ and x gives the transformation law of the gauge potential as

Maurer-Cartan form

$$\mathcal{A} \mapsto \mathcal{A}' = g^{-1}\mathcal{A}g + g^{-1}dg,$$

which is the geometric version of the gauge potential transformation law.

1.4.2 The Field Strength

Next up is the principal curvature. On the spacetime manifold this gives the *field strength* $\mathcal{F} := \phi^*\Omega$, which is a Lie algebra valued 2-form on spacetime. We once again investigate how this object changes under a local gauge transformation. We find

field strength

$$\mathcal{F} \mapsto \mathcal{F}' = \phi'^*\big(d\omega + \omega \wedge \omega\big) = d(\phi'^*\omega) + (\phi'^*\omega) \wedge (\phi'^*\omega) = d\mathcal{A}' + \mathcal{A}' \wedge \mathcal{A}'.$$

We use the transformation law for $\mathcal{A}$ and solve term by terms. First we use

$$0 = d(\mathbb{1}) = d(gg^{-1}) = (dg)g^{-1} + gd(g^{-1}) \implies d(g^{-1}) = -g^{-1}(dg)g^{-1},$$

to find

$$\begin{aligned} d\mathcal{A}' &= d(g^{-1}\mathcal{A}g) + d(g^{-1}dg) \\ &= d(g^{-1}) \wedge \mathcal{A}g + g^{-1}(d\mathcal{A})g - g^{-1}\mathcal{A} \wedge dg + d(g^{-1}) \wedge dg \\ &= -g^{-1}(dg)g^{-1} \wedge \mathcal{A}g + g^{-1}(d\mathcal{A})g - g^{-1}\mathcal{A} \wedge dg - g^{-1}(dg)g^{-1} \wedge dg \\ &= -g^{-1}(dg \wedge \mathcal{A}g) + g^{-1}(d\mathcal{A})g - g^{-1}(\mathcal{A}g \wedge dg) - g^{-1}dg \wedge g^{-1}dg. \end{aligned}$$

The second term gives

$$\begin{aligned} \mathcal{A}' \wedge \mathcal{A}' &= (g^{-1}\mathcal{A}g + g^{-1}dg) \wedge (g^{-1}\mathcal{A}g + g^{-1}dg) \\ &= g^{-1}\mathcal{A}g \wedge g^{-1}\mathcal{A}g + g^{-1}\mathcal{A}g \wedge g^{-1}dg + g^{-1}dg \wedge g^{-1}\mathcal{A}g + g^{-1}dg \wedge g^{-1}dg \\ &= g^{-1}(\mathcal{A} \wedge \mathcal{A})g + g^{-1}(\mathcal{A}g \wedge dg) + g^{-1}(dg \wedge \mathcal{A}g) + g^{-1}dg \wedge g^{-1}dg \end{aligned}$$

Combining both terms gives

$$\mathcal{F} \mapsto \mathcal{F}' = g^{-1}(d\mathcal{A})g + g^{-1}(\mathcal{A} \wedge \mathcal{A})g = g^{-1}\mathcal{F}g.$$

This highlights an important difference between the gauge potential and the field strength. The field strength transforms covariantly, just like the objects on the principal bundle. This means that the field strength is purely geometric in nature, it transforms as a pure Lie algebra element. The gauge potential however has an inhomogeneous term $gd(g^{-1})$ in its transformation law. When talking about objects on the spacetime manifold we say that $\mathcal{F}$ is adjoint *covariant* with respect to the Lie algebra, whereas we say that $\mathcal{A}$ is adjoint *affine* with respect to the Lie algebra. Even if it does not look like a pure adjoint transformation, we still say that $\mathcal{A}$ carries the adjoint representation, since the associated bundle object ω is adjoint covariant.

covariant
affine

1.5 Representations

Now that we can work with objects on our spacetime manifold $\mathcal{M}$, we wish to represent these objects in matrix form. The Lie group G, the fibre, will have a *representation* $\rho : G \mapsto GL(V)$, where V is a vector space and $GL(V)$ is the group of invertible linear transformations from V to itself (the automorphism group of V). In practice, V is often n-dimensional real or complex space, in which case $GL_n(\mathbb{R})$ or $GL_n(\mathbb{C})$ is a matrix group. A representation must respect the group structure, meaning that for all $g, h \in G$ we require that $\rho(g)\rho(h) = \rho(gh)$. The Lie algebra too has a *Lie algebra representation* $d\rho : \mathfrak{g} \mapsto \mathrm{End}(V)$, where if V is n-dimensional real or complex space, then $\mathrm{End}(V)$ is $\mathbb{R}^{n\times n}$ or $\mathbb{C}^{n\times n}$, the space of real or complex $n \times n$ matrices. The Lie algebra representation is the pushforward of the Lie group representation, that is, $d\rho(\mathfrak{X}) = \frac{d}{dt}|_{t=0}\rho(\exp(t\mathfrak{X}))$. The representation of the Lie algebra must respect the Lie algebra structure, namely for all $\mathfrak{X}, \mathfrak{Y} \in \mathfrak{g}$ we require that $[d\rho(\mathfrak{X}), d\rho(\mathfrak{Y})] = d\rho([\mathfrak{X}, \mathfrak{Y}])$.

representation

Lie algebra representation

1.5.1 Matter Fields

Using the representation $\rho : G \mapsto GL(V)$ of G we can form a new fibre bundle

$$\bar{P} := P \times_\rho V := (P \times V)/\sim$$

called the *associated bundle*. There is a projection map $\bar{\pi} : \bar{P} \mapsto \mathcal{M}, \bar{\pi}([p,v]_\sim) = x$. The equivalence relation $\sim$ is given by $(p,v) \sim (pg, \rho(g)v)$ for all $p \in P, v \in V, g \in G$. This property is called *matter covariance*. In particular, this means that the associated fibre over a point $x \in \mathcal{M}$ is $\bar{P}_x := \bar{\pi}^{-1}(x) = (P_x \times V)/G \cong V$, meaning the associated bundle is a vector bundle.

associated bundle

matter covariance

Choose a local section $\phi : \mathcal{U} \subset \mathcal{M} \mapsto P$ of the principal bundle and a local section $\bar{\phi} : \mathcal{U} \in \mathcal{M} \mapsto \bar{P}$ of the associated bundle, for a neighbourhood $\mathcal{U}$ of some $x \in \mathcal{M}$. Since elements of $\bar{P}_x$ are equivalence classes, we can represent the local section as $\bar{\phi}(x) = [p,v]_\sim \in \bar{P}_x$. We can represent any $p \in P_x \cong G$ as $p = \phi(x)h$ for some $h \in G$. Then

$$\bar{\phi}(x) = [p,v]_\sim = [\phi(x)h, v]_\sim = [\phi(x), \rho(h^{-1})v]_\sim =: [\phi(x), \Psi(x)]_\sim,$$

where $\Psi(x) := \rho(h^{-1})v \in V$ is the local *matter multiplet* field. This is a vector in V that transforms under the representation ρ of G. Now change section $\phi(x) \mapsto \phi'(x) = \phi(x)g(x)$ for some $g(x) \in P_x \cong G$, this is the local gauge transformation we saw before. We then find

matter multiplet

$$\bar{\phi}(x) \mapsto \bar{\phi}'(x) = [\phi(x)g(x), \Psi(x)]_\sim = [\phi(x), \rho(g^{-1}(x))\Psi(x)]_\sim =: [\phi(x), \Psi'(x)]_\sim,$$

where $\Psi'(x) = \rho(g^{-1}(x))\Psi(x)$ is exactly the transformation law for matter multiplet fields. In this way we see that the associated bundle is the geometry of the matter multiplet fields. By a local gauge transformation we require that the matter multiplet field $\Psi(x)$ must also transform in order to account for this transformation. Note that the infinitesimal Lie algebra action of the gauge transformation $\Psi \mapsto \rho(g^{-1})\Psi$ is $\Psi \mapsto \mathfrak{X}^\#\Psi = d\rho(\mathfrak{X})\Psi$, where $\mathfrak{X} \in \mathfrak{g}$ is the Lie algebra element that generates $g \in G$. This follows directly from the definition of $d\rho$ via the exponential map.

1.5.2 Covariant Differentiation

We see that representing the geometric section $\bar{\phi}(x)$ locally on the spacetime gives us the matter multiplet field $\Psi(x)$. However, this definition is entirely local to the point $x \in \mathcal{M}$. We have no way to compare $\Psi(x)$ with $\Psi(x + \delta x)$, since $\bar{\phi}(x)$ and $\bar{\phi}(x + \delta x)$ live in different fibres $\bar{P}_x$ and $\bar{P}_{x+\delta x}$. Usually one would measure this with the pushforward / exterior derivative $d\Psi$, but due to possible change in the gauge/fibre direction this is not well defined. Indeed it can be seen that if we naively use $d\Psi$ then under a local gauge transformation

$$d\Psi \mapsto d\Psi' = d(\rho(g^{-1}(x))\Psi) = \rho(g^{-1}(x))d\Psi + (d\rho(g^{-1}(x))\Psi.$$

It is clear that the second term destroys covariance, so we must find an alternative way to define a derivative of Ψ that allows us to relate the matter field at different spacetime points. The problem which causes the second term to appear is that exterior differentiation also gives movement in the gauge / fibre direction. We wish to have a notion of differentiation that is restricted to the horizontal direction.

Consider a vector field $\mathcal{X}(x) \in T_x\mathcal{M}$. The integral curves of the vector field connect spacetime points in $\mathcal{M}$. Since we found an isomorphism $d\pi : H_pP \mapsto \mathcal{M}$ there exists a unique $\tilde{\mathcal{X}}(p) \in H_pP$ satisfying $d\pi_p(\tilde{\mathcal{X}}) = \mathcal{X}(\pi(p))$. We call $\tilde{\mathcal{X}}(p)$ the *horizontal lift* of $\mathcal{X}(\pi(p))$ to the principal bundle. Specifically, the horizontal lift $\tilde{\mathcal{X}}(p)$ is the horizontal part of $d\phi_x(\mathcal{X})$, so by subtracting the vertical part we see that

horizontal lift

$$\tilde{\mathcal{X}}(p) = d\phi_x(\mathcal{X}) - \omega(d\phi_x(\mathcal{X}))^{\#} = d\phi_x(\mathcal{X}) - ((\phi^*\omega)_x(\mathcal{X}))^{\#} = d\phi_x(\mathcal{X}) - (\mathcal{A}_x(\mathcal{X}))^{\#}.$$

Now choose a section $\bar{\phi}(x)$ of the associated bundle that encodes the geometry of the matter field $\Psi(x)$. We can write $\bar{\phi}(x) = [p, \Psi(x)]_\sim$. Since the horizontal tangent space is independent of the fibre we find $H_pP = H_p\bar{P}$, and the horizontal lift $\tilde{\mathcal{X}}$ is well defined on the associated bundle. We can thus transport $\Psi(x) \in V$ along $\tilde{\mathcal{X}}$. We use this to define the *associated connection* $D_\mathcal{X}$ along $\mathcal{X}$ as $(D_\mathcal{X}\bar{\phi})(x) := [p, -\tilde{\mathcal{X}}(p)\Psi(x)]_\sim$. Under local trivialization this becomes

associated connection

$$\nabla_\mathcal{X}\Psi := -\tilde{\mathcal{X}}(p)\Psi = d\Psi(\mathcal{X}) + (\mathcal{A}_x(\mathcal{X}))^{\#}\Psi = d\Psi(\mathcal{X}) + d\rho(\mathcal{A}(\mathcal{X}))\Psi[!],$$

where we call $\nabla_\mathcal{X}$ the *covariant derivative* along $\mathcal{X}$. Independently of $\mathcal{X}$ and Ψ it can be written as $\nabla = d + d\rho(\mathcal{A})$. Note that in contrast to when $\mathcal{A}$ appears in the field strength $\mathcal{F}$, where $\mathcal{A}$ is in the adjoint representation, here $\mathcal{A}$ is in the representation $d\rho$ that the matter multiplet transforms under, which may or may not be the adjoint representation. From the principal bundle $\mathcal{A}$ intrinsically carries the adjoint representation, but to act on matter fields it has to pass to the associated bundle, where it picks up the representation of the matter field, so that it can act on this field.

covariant derivative

Under a gauge transformation both the covariant derivative and the matter multiplet it acts on transform. As such it makes most sense to consider this combined transformation law. We find

$$\begin{aligned}
\nabla\Psi \mapsto \nabla'\Psi' &= d(\rho(g^{-1})\Psi) + d\rho(\mathcal{A}')\rho(g^{-1})\Psi \\
&= d\rho(g^{-1})\Psi + \rho(g^{-1})d\Psi + d\rho(g^{-1}\mathcal{A}g)\rho(g)^{-1}\Psi - d\rho(g^{-1}dg)\rho(g)^{-1}\Psi \\
&= d\rho(g^{-1})\Psi + \rho(g^{-1})d\Psi + \rho(g)^{-1}d\rho(\mathcal{A})\rho(g)\rho(g)^{-1}\Psi - d\rho(g^{-1})\rho(g)\rho(g)^{-1}\Psi \\
&= \rho(g^{-1})(d + d\rho(\mathcal{A}))\Psi = \rho(g^{-1})\nabla\Psi,
\end{aligned}$$

which is to be expected since $\nabla\Psi \in V$.

1.5.3 Component Form

The final step we want to take to be ready for the physics is writing the various objects we found in *component form*. This entails choosing a basis for each space involved and decomposing the various objects in terms of these basis elements.

component form

For the Lie algebra valued objects we need to choose a representation. Since such objects are adjoint covariant they get the *adjoint representation*. The adjoint representation of G is the map $\rho_{\mathrm{ad}} : G \mapsto GL(\mathfrak{g})$ such that $\rho_{\mathrm{ad}}(g)\mathfrak{X} \mapsto g\mathfrak{X}g^{-1}$ for $g \in G$ and $\mathfrak{X} \in \mathfrak{g}$. The adjoint representation of $\mathfrak{g}$ is the map $d\rho_{\mathrm{ad}} : \mathfrak{g} \mapsto \mathfrak{gl}(\mathfrak{g}) \cong \mathrm{Aut}(\mathfrak{g})$. To get an explicit form for the Lie algebra representation we consider an element $\mathfrak{Y}$ of the Lie algebra that generates the group element g by the exponential map. We then expand the exponential map to first order for infinitesimal t to get an infinitesimal group element at the identity. The representation of this group element then identifies the representation of the Lie algebra element $\mathfrak{Y}$. Concretely we find

adjoint representation

$$\begin{aligned}
\rho_{\mathrm{ad}}(\mathbb{1} + t\mathfrak{Y})\mathfrak{X} &= (\mathbb{1} + t\mathfrak{Y} + \dots)\mathfrak{X}(\mathbb{1} - t\mathfrak{Y} + \dots) \\
&= \mathfrak{X} + t[\mathfrak{Y}, \mathfrak{X}] + \dots \\
&= (\mathbb{1} + t[\mathfrak{Y}, \cdot] + \dots)\mathfrak{X} \\
&= (\mathbb{1} + t d\rho_{\mathrm{ad}}(\mathfrak{Y}) + \dots)\mathfrak{X}.
\end{aligned}$$

Hence $d\rho_{\mathrm{ad}}(\mathfrak{Y})$ acts on the Lie algebra by taking the Lie bracket $[\mathfrak{Y}, \cdot]$.

The other instance where Lie group elements come into play is for the matter multiplet fields. Matter multiplet fields are not Lie algebra valued and they do not carry the adjoint action, so they do not transform under the adjoint representation. Instead they transform under any fundamental representation. This observation is chosen to match the physics of the gauge theory. This choice is mathematically non-unique, but unique in the sense that we wish to match experiment. These representations are just denoted ρ and $d\rho$.

In going to component form we move from mathematics convention to physics convention. The key difference is that the generators ξ that span the Lie algebra are anti-Hermitian mathematics convention, while in the physics convention one defines the generators as $T^a := i d\rho_{\mathrm{ad}}(\xi^a)$, such that the generators T^a are Hermitian and the group elements are unitary. The mathematics convention is tailored to group algebra, while the physics convention results in a unitary theory. In the physics convention we find the commutator relation $[T^a, T^b] = i f^{ab}{}_c T^c$.

The physics convention also introduces the *coupling constant* g in several places. This encodes one of the degrees of freedom in the Lagrangian that has to be fixed by experiment, as we will see later. In particular, this leads to a redefinition $A := -\mathcal{A}/g$ and $F := -\mathcal{F}/g$.

coupling constant

We will not write $d\rho_{\mathrm{ad}}(\mathfrak{X})$ for a given Lie algebra valued object $\mathfrak{X}$, instead for clarity of notation it is understood that $\mathfrak{X}$ is in the adjoint representation (for example we write $\xi = -iT$). We will however from now on write $U(x) := \rho(-g(x)) = \exp(\alpha_a(x)\xi^a) = \exp(i\alpha_a(x)T^a)$ for the group representation. The minus sign from the generator definition results effectively in a mapping $g \mapsto U^{-1}$ and $g^{-1} \mapsto U$.

Hermitian generators

spacetime basis

We choose a basis $\{\xi^a\}_{a=1}^n$ of generators for the Lie algebra, which gives the *Hermitian generators* $T^a = i\xi^a$. We also choose a local orthonormal *spacetime basis* $\{dx^\mu\}_{\mu=0}^3$ for the spacetime manifold $\mathcal{M}$. We first treat the gauge potential, now in the adjoint representation. It decomposes as

$$\mathcal{A} = -gA = -gA_{a\mu}\xi^a dx^\mu = igA_{a\mu}T^a dx^\mu.$$

This allows us to determine the field strength in component form as

$$\begin{aligned}
\mathcal{F} &= d\mathcal{A} + \frac{1}{2}[\mathcal{A}, \mathcal{A}] \\
&= ig\Big(\partial_\mu A_{a\nu} T^a dx^\mu \wedge dx^\nu + ig\frac{1}{2}A_{a\mu}A_{b\nu}[T^a, T^b] dx^\mu \wedge dx^\nu\Big) \\
&= ig\frac{1}{2}\Big(\big(\partial_\mu A_{a\nu} - \partial_\nu A_{a\mu}\big)T^a + igA_{a\mu}A_{b\nu}[T^a, T^b]\Big) dx^\mu \wedge dx^\nu \\
&= ig\frac{1}{2}\Big(\partial_\mu A_{a\nu} - \partial_\nu A_{a\mu} - gf_a{}^{bc}A_{b\mu}A_{c\nu}\Big)T^a dx^\mu \wedge dx^\nu \\
&= ig\frac{1}{2}F_{a\mu\nu}T^a dx^\mu \wedge dx^\nu = ig\frac{1}{2}F_{\mu\nu} dx^\mu \wedge dx^\nu = -gF,
\end{aligned}$$

where we used the commutator relations of the generators and relabelling of dummy indices. The covariant derivative becomes

$$\nabla = d + \mathcal{A} = \partial_\mu dx^\mu + igA_{a\mu}T^a dx^\mu = (\partial_\mu + igA_{a\mu}T^a)dx^\mu = \nabla_\mu dx^\mu.$$

We now work out the various local gauge transformation laws in component form are as follows. We do this for an infinitesimal transformation, such that we can expand $U(x) = 1 + i\alpha_a(x)T^a + \mathcal{O}(\alpha^2)$ and work to first order in α. This simplifies significantly the required calculations and allows us to derive invariance results, which will be true also for finite transformations. For A we have

$$\begin{aligned}
\mathcal{A} \mapsto \mathcal{A}' &= U\mathcal{A}U^{-1} + Ud(U^{-1}) = \Big(-gUA_\mu U^{-1} + U\partial_\mu(U^{-1})\Big)dx^\mu \\
&\approx ig\Big((1 + i\alpha_b T^b)A_\mu(1 - i\alpha_c T^c) + \frac{1}{ig}(1 + i\alpha_c T^c)\partial_\mu(1 - i\alpha_a T^a)\Big)dx^\mu \\
&\approx ig\Big(A_{a\mu}T^a + i\alpha_b T^b A_{a\mu}T^a - iA_{a\mu}T^a\alpha_c T^c - \frac{1}{g}(\partial_\mu\alpha_a)T^a\Big)dx^\mu \\
&= ig\Big(A_{a\mu}T^a + i\alpha_b A_{a\mu}[T^b, T^a] - \frac{1}{g}(\partial_\mu\alpha_a)T^a\Big)dx^\mu \\
&= ig\Big(A_{a\mu}T^a - \alpha_b A_{a\mu}f^{ab}{}_c T^c - \frac{1}{g}(\partial_\mu\alpha_a)T^a\Big)dx^\mu \\
&= ig\Big(A_{a\mu} - \frac{1}{g}\partial_\mu\alpha_a - \alpha_b A_{c\mu}f_a{}^{bc}\Big)T^a dx^\mu = igA'_{a\mu}T^a dx^\mu,
\end{aligned}$$

such that for the component we find the transformation law

$$A_{a\mu} \mapsto A'_{a\mu} = A_{a\mu} - \frac{1}{g}\partial_\mu\alpha_a - \alpha_b A_{c\mu}f_a{}^{bc}.$$

For the field strength we simply follow the same derivation as the adjoint part of $\mathcal{A}'$. We find

$$\mathcal{F} \mapsto \mathcal{F}' = ig\frac{1}{2}\Big(F_{a\mu\nu} - \alpha_b F_{c\mu\nu} f_a{}^{bc}\Big) T^a dx^\mu \wedge dx^\nu$$

such that

$$F_{a\mu\nu} \mapsto F'_{a\mu\nu} = F_{a\mu\nu} - \alpha_b F_{c\mu\nu} f_a{}^{bc}.$$

The covariant derivative transforms as

$$\nabla\Psi \mapsto \nabla'\Psi' = U\nabla_\mu \Psi dx^\mu \quad \text{such that} \quad \nabla_\mu \Psi \mapsto \nabla'_\mu \Psi' = U\nabla_\mu \Psi.$$

We now have all the individual components that make up the Lagrangian. We do not however have all the techniques to combine them into a well defined Lagrangian that gives proper equations of motion. These techniques will be developed in the next two chapters.

1.6 The Dual Role of Space

The spacetime manifold $\mathcal{M}$ takes a special role in the construction of our theory. It is the object on which the physics lives. The gauge symmetries we have seen up until now do not really act as a symmetry of the manifold itself, but are contained within it through the fibre construction. As such, we call these *internal symmetries.* However, the spacetime manifold also has symmetries that act on the manifold itself, we call these *external symmetries.* For flat space these are the symmetries of the (proper) *Poincaré group*, which includes translations, rotations, and boosts. In a curved spacetime these symmetries are not global any more, but they do remain as local symmetries of the (proper) *Lorentz group* (Poincaré without translations), since we can always form a local inertial frame, such that locally space looks flat. For now we will remain in flat space, postponing the treatment of curved space to Chapter 3.

internal symmetries

external symmetries

Poincaré group

Lorentz group

1.6.1 The Frame Bundle

Let us investigate deeper the spacetime manifold $\mathcal{M}$. At every point $x \in \mathcal{M}$ there exists a tangent space $T_x\mathcal{M}$, which is a vector space. A *frame* of $\mathcal{M}$ at x is an ordered orthonormal basis $(e_\alpha)_{\alpha=0}^3 = (e_0, e_1, e_2, e_3)$ of $T_x\mathcal{M}$ that consists of four orthonormal basis vectors of the vector space $T_x\mathcal{M}$. There are an infinite number of possible frames. Let $\mathcal{B}_x$ be the set of all frames of $T_x\mathcal{M}$. If we take $\mathcal{B}_x$ to be a fibre over $x \in \mathcal{M}$, then we form a fibre bundle $\mathcal{B}$ called the *frame bundle* of $\mathcal{M}$. There is a projection map $\pi_\mathcal{B} : \mathcal{B} \mapsto \mathcal{M}$. In terms of the metric g on $\mathcal{M}$, the condition of orthonormality can be formulated as follows. Let $(e_\alpha)_{\alpha=0}^3 \in \mathcal{B}_x$, then this frame should satisfy $g(e_\alpha, e_\beta) = \eta_{\alpha\beta}$, where $\eta_{\alpha\beta}$ are the components of the flat space metric. Given a coordinate basis $\{\partial_\mu\}_{\mu=0}^3 := \{\partial/\partial x^\mu\}_{\mu=0}^3$ we can write basis elements of the tangent space as $e_\alpha = e_\alpha^\mu \partial_\mu$, where e_α^μ is called the *tetrad.*

frame

frame bundle

tetrad

Changing to a different frame goes by a special orthogonal transformation $e_\alpha \mapsto e'_a = \Lambda^\beta{}_\alpha e_\beta$. The group of such transformations is the Lorentz group $SO^+(1,3)$, which encompasses continuous boosts and rotations. The transformation happens by means of a right group action. This means that the frame bundle constitutes a principal bundle with symmetry group $SO^+(1,3)$. The group $SO^+(1,3)$ describes how scalars, vectors, and tensors transform. It is the subgroup of $SO(1,3)$ with positive determinant. In the same way there exists $SO^-(1,3) \subset SO(1,3)$, which is the subgroup with negative determinant. The two subgroups are exactly the two disconnected components of $SO(1,3)$, and they are isomorphic to each other. This structure hints to the fact that there exists a *double cover* of $SO^+(1,3)$. The double cover connects the $SO^+(1,3)$ component, which contains the identity $\mathbb{1}$ to the $SO^-(1,3)$ component, which contains $-\mathbb{1}$. This means that after an integer number of full 2π rotations we may end up at $-\mathbb{1}$ instead of $\mathbb{1}$. This is of course the structure we expect for particles with half integer spin: matter fermions. As such, the universal cover group of $SO^+(1,3)$ is called the *spin group* and is denoted $Spin(1,3)$.

double cover

spin group

There exists a group homomorphism $\varphi : Spin(1,3) \mapsto SO^+(1,3)$ called the *cover map* of $Spin(1,3)$ onto $SO^+(1,3)$. By the fact that φ is a group homomorphism, we know that the identity in $Spin(1,3)$ is mapped to the identity in $SO^+(1,3)$. Since the cover space locally looks the same as the base space, the tangent spaces at the identity look the same, so $Spin(1,3)$ has the same Lie algebra as $SO^+(1,3)$, that is, $\mathfrak{so}(1,3)$. Let us investigate this Lie algebra further.

cover map

1.6.2 The Spin Bundle

The Lie algebra $\mathfrak{so}(1,3)$ has six generators, three for rotations around e_1, e_2, and e_3 and three for boosts in the direction of e_1, e_2, and e_3. The boosts are essentially hyperbolic rotations in the e_0-e_1, e_0-e_2, and e_0-e_3 plane. We will consider the generators directly in physics convention, where they carry an extra factor of i to make them hermitian. The six generators can be combined into a 4×4 anti-symmetric matrix $M^{\alpha\beta} = -M^{\beta\alpha}$, which indeed has six independent components.

The generator matrix $M^{\alpha\beta}$ satisfies the commutation relation

$$[M^{\alpha\beta}, M^{\gamma\delta}] = i(\eta^{\beta\gamma}M^{\alpha\delta} - \eta^{\alpha\gamma}M^{\beta\delta} - \eta^{\beta\delta}M^{\alpha\gamma} + \eta^{\alpha\delta}M^{\beta\gamma}).$$

We get the three rotation generators $\{J^i\}_{i=1}^3$ an three boost generators $\{K^i\}_{i=1}^3$ as

$$J^i = \frac{1}{2}\epsilon^i{}_{jk}M^{jk}, \quad K^i = M^{0i}, \quad i,j,k \in \{1,2,3\},$$

which satisfy the commutation relations

$$[J^i, J^j] = i\epsilon^{ij}{}_k J^k, \quad [J^i, K^j] = i\epsilon^{ij}{}_k K^k, \quad [K^i, K^j] = -i\epsilon^{ij}{}_k J^k,$$

where $\epsilon^{ij}{}_k$ is the totally anti-symmetric symbol in three dimensions.

The Lie algebra $\mathfrak{so}(1,3)$, as a four dimensional vector space, can be decomposed into the direct product of two other Lie algebras, each two dimensional vector spaces. To see how this comes about, we construct the generators for these two other Lie algebras in terms of the generators of $\mathfrak{so}(1,3)$ we just found. They are

$$S_L^i = \frac{1}{2}(J^i + iK^i), \quad S_R^i = \frac{1}{2}(J^i - iK^i),$$

where L and R suggestively stand for "left" and "right", the meaning of which shall become clear later. Using the commutators of the J's and K's we find that the commutators for the S's are

$$[S_L^i, S_L^j] = i\epsilon^{ij}{}_k S_L^k, \quad [S_R^i, S_R^j] = i\epsilon^{ij}{}_k S_R^k.$$

This is the commutation relation of the Lie algebra $\mathfrak{su}(2)$, related to the *special unitary group* $SU(2)$ of unitary matrices with unit determinant. We thus find that

special unitary group

$$\mathfrak{so}(1,3) = \mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R.$$

This means that the spin group has the Lie algebra structure of $\mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R$. This motivates the forming of a principal bundle $\mathcal{S}$ called the *spin bundle* of $\mathcal{M}$ with as fibre the spin group $Spin(1,3)$. There is a projection map $\pi_{\mathcal{S}} : \mathcal{S} \mapsto \mathcal{M}$.

spin bundle

Since the generator matrix $M^{\alpha\beta}$ combines the basis of $\mathfrak{so}(1,3)$ into one object, it itself can be used as a basis for the Lie algebra. In fact, given the *spin connection* $\omega_{\mathcal{S}}$ on the spin bundle, it can be written in component form as

spin connection

$$\omega_{\mathcal{S}} = -\frac{1}{2}\omega_{\mu\alpha\beta}M^{\alpha\beta}dx^\mu,$$

where the factor $1/2$ and the absence of a factor i, compared to the gauge connections we saw before, are by convention. Depending on the representation of the Lie algebra, the representation of the generators $\Sigma^{\alpha\beta}$ of $M^{\alpha\beta}$ will change. The covariant derivative for the spin bundle would look like

$$\nabla_\mu = \partial_\mu + \frac{1}{2}\omega_{\mu\alpha\beta}\Sigma^{\alpha\beta}.$$

However, we will see in chapter 3 that the components $\omega_{\mu\alpha\beta}$ of the spin connection $\omega_{\mathcal{S}}$ can be related to the components $\Gamma^\nu_{\mu\rho}$ of the frame connection $\omega_{\mathcal{B}}$ using the tetrad as

$$\omega_{\mu\alpha\beta} = e_{\alpha\nu}(\partial_\mu e^\nu_\beta + \Gamma^\nu_{\mu\rho}e^\rho_\beta).$$

In flat space we can choose coordinates such that $g_{\mu\nu} = \eta_{\mu\nu}$ and $\Gamma^\nu_{\mu\rho} = 0$ meaning that $e^\mu_\alpha = \delta^\mu_\alpha$ and $\partial_\mu e^\nu_\beta = 0$, meaning $\omega_{\mu\alpha\beta} = 0$ and the partial derivative does not get an additional term from the spin bundle. This means that in flat space the spin bundle is a *trivial bundle*, which is a consequence of the fact that we can choose the same local inertial frame on the whole manifold $\mathcal{M}$. The spin group gives a global symmetry. However, this does not mean that $\omega_{\mu\alpha\beta} = 0$ for all coordinates on flat space, we are only guaranteed to be able to choose a set of coordinates for which this is true. For example, in polar coordinates on flat space $\Gamma^\nu_{\mu\rho} \neq 0$, so also $\omega_{\mu\alpha\beta} \neq 0$.

trivial bundle

1.6.3 The Irreducible Representations of $\mathfrak{su}(2)$

We now wish to find the *irreducible representations* of the Lie algebra, which will turn out to be

irreducible representations

$$(j_L, j_R) := d\rho_{(j_L, j_R)} : \mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R \mapsto \mathfrak{gl}(\mathbb{C}^4).$$

These are representations that cannot be further decomposed into other representations, making them irreducible. They are labelled by two labels j_L and j_R, since the irreducible representations of a single $\mathfrak{su}(2)$ are labelled by a single j, as we will see now.

1.6.4 Spin Multiplets

1.7 The Lagrangian

1.7.1 The Matter Lagrangian

1.7.2 The Gauge Lagrangian

1.7.3 The Full Lagrangian and Equations of Motion

1.8 Symmetry Breaking

Chapter 2

The Standard Model of Particle Physics

Chapter 3

Gravity

Appendix A

The Second Cartan Structure Equation

The principal curvature is defined as as $\Omega(X, Y) := -\omega([\tilde{\mathcal{X}}, \tilde{\mathcal{Y}}])$. We would like to rewrite this into a form that is independent of the vector fields X and Y used to construct the infinitesimal loop that let us define this quantity. Using the definition of the horizontal projection and the bi-linearity of the commutator bracket we obtain

$$\Omega(X, Y) = -\omega([X, Y]) + \omega([X, \omega(Y)^{\#}]) + \omega([\omega(X)^{\#}, Y]) - \omega([\omega(X)^{\#}, \omega(Y)^{\#}]).$$

To treat the mixed terms we use the adjoint covariance of the connection to show that

$$\omega(X(pg)) = \omega(\tilde{\mathcal{X}}(pg)) + \omega(\mathfrak{X}^{\#}(pg)) = \omega((g^{-1}\mathfrak{X}g)^{\#})(p) = g^{-1}\omega(X)g.$$

We use this property with $g = \omega(Y)^{\#}$ and to derive that

$$\begin{aligned}
\omega(Y)^{\#}\omega(X)(p) &= \frac{d}{dt}\Big|_{t=0} \exp(-t\omega(Y))\omega(X)(p)\exp(t\omega(Y)) \\
&= \omega(Y)\omega(X)\big(\exp(t\omega(Y))p\big) - \omega(X)\big(\exp(t\omega(Y))p\big)\omega(Y)\Big|_{t=0} \\
&= [\omega(Y), \omega(X)](p).
\end{aligned}$$

Using this we can evaluate the mixed terms as

$$\omega([X, \omega(Y)^{\#}]) = X\omega(\omega(Y)^{\#}) + \omega(Y)^{\#}\omega(X) = X\omega(Y) + [\omega(X), \omega(Y)],$$

and

$$\omega([\omega(X)^{\#}, Y]) = \omega(X)^{\#}\omega(Y) + \omega(Y)\omega(\omega(X)^{\#}) = -Y\omega(X) - [\omega(X), \omega(Y)].$$

Using the antisymmetry of the commutator bracket and swapping X and Y we get a similar result for the other mixed term. Combining all we have found together we get

$$\Omega(X,Y) = -\omega([X,Y]) + X\omega(Y) - Y\omega(X) - \omega([\omega(X)^{\#}, \omega(Y)^{\#}]).$$

We recognize the exterior derivative of the connection 1-form as

$$d\omega(X,Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y]),$$

which results in

$$\Omega(X,Y) = d\omega(X,Y) - \omega([\omega(X)^{\#}, \omega(Y)^{\#}]).$$

We finally show that

$$\begin{aligned}
&[\omega(X)^{\#}, \omega(Y)^{\#}](p) = \\
&\quad = \frac{\partial^2}{\partial s \partial t}\Big|_{s=t=0} p \exp(s\omega(X)) \exp(t\omega(Y)) \exp(-s\omega(X)) \exp(-t\omega(Y)) \\
&\quad = \frac{\partial^2}{\partial s \partial t}\Big|_{s=t=0} p(1 + s\omega(X) + \mathcal{O}(s^2)) \exp(t\omega(Y)) \exp(-s\omega(X)) \exp(-t\omega(Y)) \\
&\quad = \frac{\partial^2}{\partial s \partial t}\Big|_{s=t=0} p(1 + s\omega(X) + t\omega(Y) + st\omega(X)\omega(Y) + \mathcal{O}(s^2, t^2)) \exp(-s\omega(X)) \exp(-t\omega(Y)) \\
&\quad = \frac{\partial^2}{\partial s \partial t}\Big|_{s=t=0} p(1 + t\omega(Y) + st(\omega(X)\omega(Y) - \omega(Y)\omega(X)) + \mathcal{O}(s^2, t^2)) \exp(-t\omega(Y)) \\
&\quad = \frac{\partial^2}{\partial s \partial t}\Big|_{s=t=0} p(1 + st[\omega(X), \omega(Y)] + \mathcal{O}(s^2, t^2)) \\
&\quad = -[\omega(X), \omega(Y)]^{\#}(p),
\end{aligned}$$

where we use the fact that we evaluate at $s = t = 0$ to disregard higher order terms. Using that

$$[\omega, \omega](X,Y) = [\omega(X), \omega(Y)] - [\omega(Y), \omega(X)] = 2[\omega(X), \omega(Y)] = 2(\omega \wedge \omega)(X,Y),$$

we get

$$\Omega(X,Y) = d\omega(X,Y) + \frac{1}{2}[\omega, \omega](X,Y) = d\omega(X,Y) + (\omega \wedge \omega)(X,Y).$$

This equation can be written independently of the vectors X and Y as

$$\Omega = d\omega + \frac{1}{2}[\omega, \omega] = d\omega + \omega \wedge \omega,$$

and is called the second Cartan structure equation.